Modification of Pierce's Classical Theory of Traveling-Wave Tubes

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***Abstract*—For a realistic case of an electron beam interacting with the electromagnetic fields supported by a thin perfectly conducting tape helix, we derive the exact dispersion relation, from which an unambiguous determination of Pierce's space-charge parameter *Q* is made. In the process of doing so, we discover that the circuit mode in the equivalent three-wave theory of Pierce must be modified at high beam current, an aspect not included in Pierce's original analysis. We quantify this circuit mode modification by a new parameter that we call *q*, introduced here for the first time in traveling-wave tube theory. In a realistic example, we find that the effect of *q* is equivalent to a modification of the circuit phase velocity by as much as two percent, which is a significant effect—equivalent to a detune of two percent.**

***Index Terms*—Dispersion relation, Pierce parameters, space-charge effect, traveling-wave tubes.**

I. Introduction

PIERCE'S classical theory of traveling-wave tubes (TWTs) laid the foundation for the generation of coherent radiation from the interaction of an electron beam with a periodic structure. Amplification of a signal of frequency ω is described by the complex wavenumber β that is a solution of the Pierce dispersion relation, which, in its most basic form [1]–[3], is a third-degree polynomial for $\beta(\omega)$,

$$\left[(\beta-\beta_e)^2-4\beta_e^2QC^3\right]\left[\beta-\beta_{ph}\right]=-\beta_e^3C^3 \quad (1)$$

where $\beta_e=\omega/v_0$, $\beta_{ph}=\omega/v_{ph}$, v_0 is the DC beam velocity, and v_{ph} is the phase velocity of the circuit wave on the cold-tube circuit. It should be noted that Pierce's original dispersion relation was a fourth-degree polynomial where the fourth root for β represented the backward propagating circuit wave; in this letter, we will use the classical forward wave gain analysis and only consider the three forward propagating waves. Equation (1) quantifies the coupling of the beam mode [first bracket on the LHS, which includes the so-called "space-charge effect" represented by Q] with the cold-tube circuit mode [second bracket, assuming zero circuit loss]. The quantity C^3 on the RHS measures the strength of the coupling between the beam and the circuit; it is proportional to the DC beam current [1]. C is known as Pierce's gain parameter. While there have been approximate models of Q for a TWT [4]–[8], no general calculation of Q is given in the literature [9].

An accurate determination of Q is important for at least three reasons: first, a small discrepancy in Q can lead to a large change in the predicted small-signal gain. Second, an accurate model for the shielding effect of the circuit on the space-charge electric field acting on a bunched beam (which depends on the value of Q in small-signal theory) is required by nonlinear TWT simulation codes such as CHRISTINE [5, Sec. II C] in order to compute large-signal quantities such as saturated output power and efficiency. Third, in Johnson's classical theory for the onset of backward wave oscillations in TWTs [10], the threshold conditions depend only on QC and on d, Pierce's loss parameter [2], [10], which implies that prediction of BWO threshold current requires accurate values of QC and d.

II. Modification of Pierce's Classical TWT Theory

We consider a realistic model of a helix circuit [11], consisting of an infinitesimally thin tape helix of radius a, pitch p, and width w, surrounded by a stratified dielectric and an outer perfectly conducting metallic cylinder of radius b (Fig. 1). The pitch angle ψ of the helix is defined by $\cot\psi\equiv k_Ha$, where $k_H\equiv 2\pi/p$. A stratified (layered) dielectric is included between the outer cylinder and the helix in order to represent the dielectric effects of the support rods in an approximate way. It has been shown (see [12], [13] of [11]) that this model is a good approximation to an actual helix TWT circuit without dispersion control elements (vanes).

An exact dispersion relation for this helix in the absence of a beam was obtained in [11] which may be solved numerically for β given the signal frequency ω. We assume as in [11] that there is no cold-tube loss ($d=0$) in this letter.

We include a mono-energetic nonrelativistic electron beam [12], of voltage V_b, current I_0, radius R_b and uniform density n_0 centered on the axis of the helix and confined by an infinite axial magnetic field so that electrons move only axially. Outside the beam region, the calculation in [11] for the vacuum electromagnetic fields carries over. Within the beam region, we calculate the small-signal AC current and the electromagnetic fields [12]. The exact hot-tube dispersion

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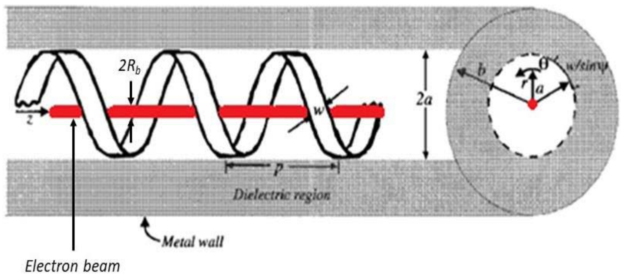

Fig. 1. Schematic diagram of tape helix TWT model.

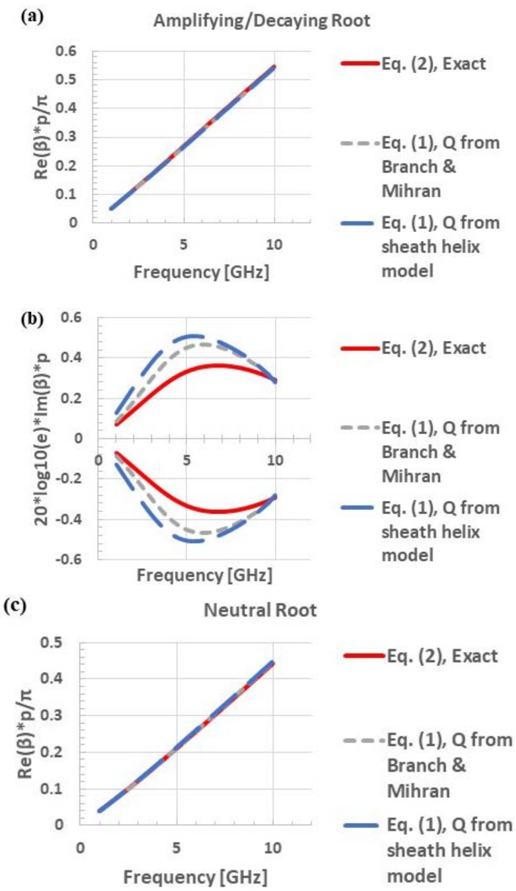

Fig. 2. Roots of the hot-tube dispersion relation, Eq. (2), for a tape helix TWT, along with the soluions of Eq. (1) for different models of Q. The quantity plotted in (a) and (c) is the phase shift per period/π; the quantity plotted in (b) is the growth or attenuation in dB per period.

relation, very similar to that in [11], is next obtained. The details of the derivation are given in [12, Appendix B].

The complicated, exact hot-tube dispersion relation must be solved numerically for β, given the frequency ω. However, for all examples that we have tried, we find that the exact hot-tube dispersion relation always admits three roots for β; two of which (β_1, β_3) are complex conjugates (corresponding to spatial decay and growth, respectively) and one of which (β_2) is purely real. See Fig. 2. Thus, the exact dispersion relation may be represented in a three-wave theory as:

$$(\beta - \beta_1)(\beta - \beta_2)(\beta - \beta_3) = 0. \tag{2}$$

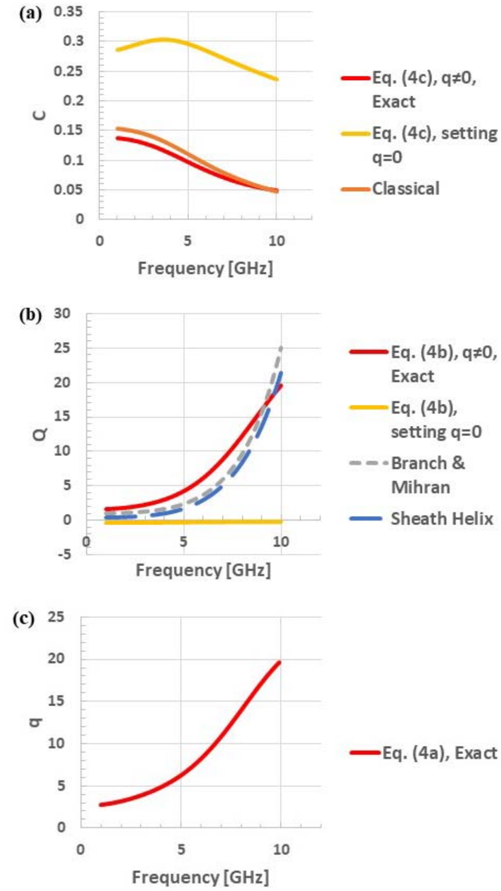

Fig. 3. Plots of the *exact* Pierce parameters [(a) C, (b) Q, (c) q] and their traditional definitions as a function of frequency.

An equivalent form of Eq. (2) may be written in a form similar to Eq. (1),

$$\left[(\beta - \beta_e)^2 - 4\beta_e^2 QC^3\right]\left[(\beta - \beta_{ph}) - 4\beta_{ph} qC^3\right] = -\beta_e^3 C^3, \tag{3}$$

where q is a new parameter introduced to account for the space-charge effect on the circuit mode. We find that this new term is essential in order to make Eqs. (2) and (3) equivalent. Note from Eq. (3) that q produces a circuit phase velocity change due to a space-charge effect, by a fraction equal to $4qC^3$. This new parameter, q, is introduced here for the first time in the literature of TWTs. Physically, Eq. (3) describes the modification of the beam mode at high current through Q, and the modification of the cold-tube circuit mode through q. In terms of the numerically obtained roots β_1, β_2, and β_3 of Eq. (2), the parameters qC^3, QC^3, and C^3 may be obtained by equating coefficients of β^2, β^1, and β^0 in Eqs. (2) and (3),

$$qC^3 = \frac{1}{4}\left(\frac{\beta_1 + \beta_2 + \beta_3 - 2\beta_e}{\beta_{ph}} - 1\right), \tag{4a}$$

$$QC^3 = \frac{1}{4}\left(1 - \frac{\beta_1\beta_2 + \beta_2\beta_3 + \beta_1\beta_3 - 2\beta_e\beta_{ph}(1 + 4qC^3)}{\beta_e^2}\right), \tag{4b}$$

$$C^3 = \frac{\beta_e^2\left(1 - 4QC^3\right)\beta_{ph}(1 + 4qC^3) - \beta_1\beta_2\beta_3}{\beta_e^3}. \tag{4c}$$

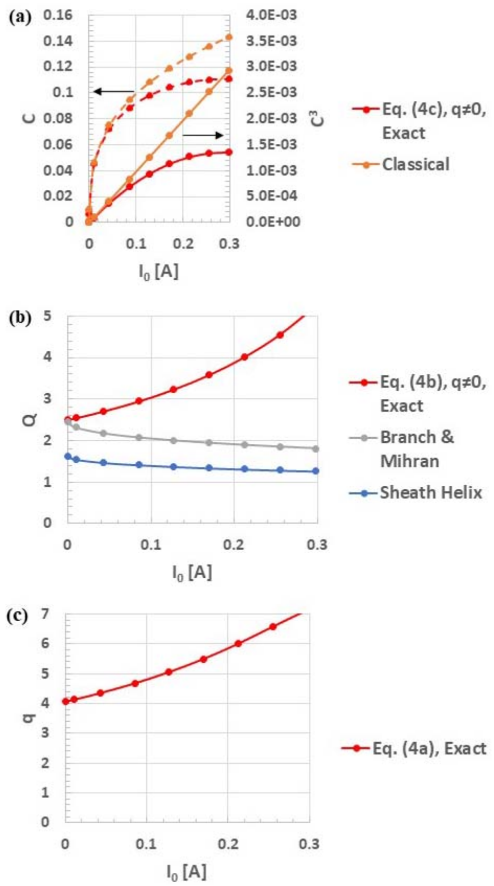

Fig. 4. Plots of the Pierce parameters [(a) C, (b) Q, (c) q] at 4.5 GHz, and their traditional definitions as functions of DC beam current. Also shown in (a) is the plot of C^3.

Numerical solutions for C, Q, and q will be presented next, and compared with previous, approximate theories.

III. Numerical Examples

We consider the first test case in [11] with $a = 0.12446$ cm, $p = 0.080137$ cm, $\psi = 5.851$ deg, $w = 0.0159$ cm, $b = 0.2794$ cm, and dielectric constant of the supporting layer $\varepsilon_r^{(2)} = 1.25$. For the electron beam, we set $V_b = 3.0$ kV, $I_0 = 0.17$ A, and $R_b = 0.05$ cm.

Figure 2 shows the numerical solutions of the exact hot-tube dispersion relation as functions of frequency along with the solutions of Pierce's classical three-wave dispersion relation, Eq. (1), with different models of Q (Branch and Mihran [4] and sheath helix [5]).

The *exact* values of the Pierce parameters obtained from Eq. (4) are plotted vs. frequency in Fig. 3, along with results using various previous models of the Pierce parameters. In Fig. 3a the 'classical' value of C is obtained from $C = (KI_0/4V_b)^{1/3}$, where K is the Pierce impedance, averaged over the beam cross-section [1], [2]. From the hot-tube results, we see that the new q parameter is required in order to obtain improved results for the Pierce parameters. For example, we see in Fig. 3a that the value of C obtained from Eq. (4c) by setting $q = 0$ is at least a factor of 2 too large. Likewise, Fig. 3b shows that the value of Q is too small if we set $q = 0$ in Eq. (4b). The values of C, Q, and q at 4.5 GHz as functions of the beam current are plotted in Fig. 4(a,b,c). Note that both Q and q depend on the beam current. When the beam current is zero, the values of C, Q, and q become undefined using Eq. (4). However, when the beam current is small, the fundamental beam and circuit modes become decoupled (c.f. Eq. (1) or (3)) and each mode propagates independently of the other. One can determine the limit as $I_0 \to 0$.

For $q = 5$ and $C = 0.1$ (Fig. 4), we have $4qC^3 = 0.02$, which is equivalent to a change of circuit phase velocity of two percent according to Eq. (3). This is equivalent to a detuning between the beam velocity and the circuit phase velocity of two percent which is very significant.

In other TWT examples for which an exact hot-tube dispersion relation is available, we find that Q is due to higher-order (non-synchronous) circuit modes [9], [12], as originally suggested by Pierce [1], and that q is due to higher-order beam modes (the non-synchronous spatial harmonics of the beam) [12]. For the present tape-helix TWT, the complexity in the formulation prevents a similar identification of the physical origins of Q and q. We anticipate that a finite value of q is generally required for the accurate representation of the dispersion relation of an electron beam interacting with the electromagnetic fields of any periodic structure.

IV. Conclusion

From a study of the numerical solutions to the exact dispersion relation for a beam interacting with the electromagnetic fields of a tape helix, we have concluded that the phase velocity of the circuit mode is affected by the beam current. This effect may be represented by a new parameter q defined in Eq. (4), and it is equivalent to a change of circuit phase velocity by the fraction of $4qC^3$.

The space-charge parameters Q and q appear symmetrically in Eq. (3) as modifications of the beam mode and the circuit mode, respectively. The present letter introduces the 'q' term and demonstrates that it is important for the tape helix TWT. In other geometries such as a disk-loaded rod slow-wave structure driven by an annular electron beam, an exact analytical form of q (and the other Pierce parameters) was obtained [12]. A term like q is also included implicitly in the formulation in [13, Appendix C], though it is not explicitly indicated. In both cases, it is shown that q is due to the beam-loading on the circuit and originates from higher-order (non-synchronous) spatial harmonics of the beam mode. Furthermore, it is also shown that q depends on the total current as opposed to the current density because it is the current that enters the exact dispersion relations (a sheet beam model was used for both cases [12], [13] where the current density is infinite, beam thickness approaches zero, and so the product is finite). More investigation into the physical origins of the 'q' term is warranted.

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