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Absolute instability and transient growth near the band edges of a traveling wave tube

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An electron beam in the slow wave structure of a traveling wave tube may be subjected to absolute instability at the lower and upper band edges, where the group velocity is zero. From a careful re-examination of the immediate vicinity of these band edges, we use the Briggs-Bers criterion to show that, contrary to previous findings, an absolute instability may arise at the lower band edge if the beam current is sufficiently high, even if the beam mode intersects with the circuit mode with a positive group velocity. However, the upper band edge was found to be more susceptible to absolute instability than the lower band edge. The threshold condition for the onset of absolute instabilities is derived analytically at both band edges. The Green's function shows possible transient temporal growth with an exponentiation rate proportional to $t^{1/3}$, whether or not the band edge is subject to an absolute instability. *Published by AIP Publishing.* https://doi.org/10.1063/1.5028385

I. INTRODUCTION

The collective interaction of an electron beam with a periodic structure is central to the generation of coherent radiation, from GHz to THz and beyond.[1] It is also an important consideration for electron beam stability in RF linacs and induction accelerators.[2] In a periodic structure, the edges of the pass band of the structure modes exhibit zero group velocity in the dispersion diagram, which is an ω vs k plot, where ω is the frequency and k is the axial wave number.[3] When the beam mode, $\omega = kv$, where v is the electron velocity of the beam, intersects with the structure mode near a band edge, band edge oscillation may result from an instability.[4–6] For a traveling wave tube (TWT) amplifier, these unwanted oscillations are to be avoided. On the other hand, these band edge oscillations have also been considered as a source of coherent radiation.[7,8]

The circuit mode of a coupled cavity TWT is shown in Fig. 1, whose lower and upper band edges are labeled as Points A and B, respectively. The beam mode, $\omega = kv$, intersects with the circuit mode in a forward wave amplifier at operating point Q, which lies between A and B. Potential excitation of absolute instability at band edges A and B for such a forward wave amplifier have been studied by Kuznetsov *et al.*,[5] and by Hung *et al.*[6] using the Briggs-Bers criterion.[9,10] Both papers reported that, for an operating point Q, an absolute instability can occur at the upper band edge but not at the lower band edge. Since the lower band edge was (erroneously) perceived as to be free of absolute instability, we attempted to assess the possible transient growth at the lower band edge. In this re-examination of the lower band edge, we discovered that the lower band edge does suffer from absolute instability when the beam current is sufficiently high, contrary to the earlier findings. This paper reports this revised study. In addition, we present the newly derived stability criterion for the onset of absolute instability at both band edges, from which we conclude that the upper band edge is more susceptible to absolute instability than the lower band edge. We also show transient temporal growths with an exponentiation rate proportional to $t^{1/3}$, whether or not the band edge is subject to an absolute instability.

Transient growth at a fractional power of time is an interesting characteristic at the band edges, a possibility suggested by Hung *et al.*[6] A zero group velocity at the band edge means that, electromagnetically, a unit in a periodic structure is isolated from its neighbor. Information is carried only by the beam. This is precisely the condition in the formulation of the cumulative beam breakup instability, originally studied by Panofsky and Bander for RF linacs,[11] which was extended to high current induction accelerators by Neil *et al.*,[12] and to linear colliders by Chao *et al.*[13] All of these authors found the fractional power of growth in time, and their main results may all be recovered in a unified analysis by assuming a zero group velocity in the structure mode in a mode-coupling analysis.[14] Note that instability whose amplitude exponentiates at a fractional power of time (at a fixed position) is not covered by the Briggs-Bers criterion,[9,10] which only governs the existence of instability that exponentiates as a linear function of time. Thus, our study provides a first demonstration of transition from exponential growth at

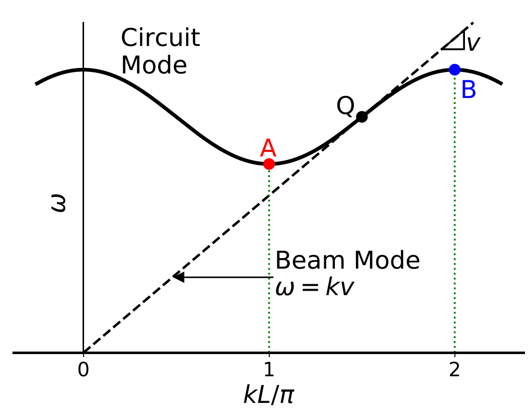

FIG. 1. The lower band edge (A), the upper band edge (B), and the operating point (Q) at which the beam mode intersects with the circuit mode in a coupled cavity TWT. *kL* is the phase shift per period.

the fractional power of time to simple exponential growth when the band edge suffers from absolute instability, and from exponential growth at the fractional power of time to stabilization when the band edge is free from absolute instability. In fact, it is this interesting transition, not covered by the Briggs-Bers criterion, which prompted our re-examination of the band edge oscillations in the first place.

We should stress that the effects of end reflections are ignored in this paper. That is, we assumed that the system is infinitely long, as in the Briggs-Bers criterion. This is a serious limitation for TWT stability analysis because the circuits are poorly matched at a band edge.

Section II presents the model and the stability criterion for the onset of absolute instability at the lower and upper band edges, A and B (Fig. 1). There, we show that the upper band edge is more susceptible to absolute instability than the lower band edge. Section III presents the Green's function constructed from the dispersion relation. Its asymptotic form shows the transition from transient exponentiation at the fractional power of time to simple exponential growth (to decay) when the band edge is (is not) subject to an absolute instability. Section IV presents the concluding remarks. The details of the derivations are given in the Appendix.

II. EXISTENCE OF ABSOLUTE INSTABILITY

For interactions sufficiently close to the band edges, either at A or at B, the circuit mode dispersion relation can be well approximated as a hyperbola in the ω-k plane in the vicinity of A or B (Fig. 1). For a wave-like perturbation of the form $e^{i\omega t-ikz}$, the hot tube dispersion relation, near A or B, may then be represented as[6]

$$D(\omega,k)\equiv(\omega-kv)^2\Big[(\omega-\omega_m)^2+2\Delta(\omega-\omega_m)-r^2(k-k_m)^2\Big]-\omega_m^4\epsilon=0\,, \tag{1}$$

where $(\omega_m,\,k_m)$ designates the band edge A or B, r and Δ are the fitting parameters for the circuit mode dispersion relation at A or B, and ϵ is the dimensionless coupling constant between the beam mode ($\omega=kv$) and the circuit mode which is represented by the square bracket of Eq. (1). Note that in Ref. 6, ω_m is defined as the focus of the hyperbola that represents the circuit mode; this focus is $\omega_m-\Delta$ in the present paper. For the lower (upper) band edge A (B), both Δ and ϵ are positive (negative).[6] Note that the dimensionless coupling constant ϵ is proportional to the beam current. It remains finite at the band edges in a careful analysis. Away from the band edges, its magnitude is approximately equal to $2C^3$, where C is Pierce's gain parameter in a TWT.[3,15,16]

We next applied the Briggs-Bers criterion[9,10] to the dispersion relation, Eq. (1). With $\omega=\omega_m(1+y)$ and $k=k_m(1+x)$, Eq. (1) is non-dimensionalized to read

$$D(x,y)\equiv(y-ux+1-u)^2\left(y^2+2\delta y-\rho^2x^2\right)-\epsilon=0, \tag{2}$$

where $u=k_mv/\omega_m$, $\delta=\Delta/\omega_m$, and $\rho=k_mr/\omega_m$. The beam mode and the circuit modes are shown in Fig. 2(a) for the lower band edge and in Fig. 2(b) for the upper band edge. In these plots, and in the numerical examples below, we used the parameters in the study by Hung *et al.*[6] For the lower band edge, $\Delta/2\pi=7.365\,\mathrm{GHz}$, $r=8.6973\times10^9\,\mathrm{m/s}$, $k_m=1.78\,\mathrm{mm}^{-1}$, and $\omega_m/2\pi=24.24\,\mathrm{GHz}$. For the upper band edge, $\Delta/2\pi=-7.365\,\mathrm{GHz}$, $r=8.6973\times10^9\,\mathrm{m/s}$, $k_m=3.560\,\mathrm{mm}^{-1}$, and $\omega_m/2\pi=36.96\,\mathrm{GHz}$.

To apply the Briggs-Bers criterion on the existence of absolute instability, we first solve the system[9,10]

$$\begin{aligned} D(x,y)&=0\\ \frac{\partial D}{\partial x}&=0, \end{aligned} \tag{3}$$

which will yield eight pairs of solutions (x_s,y_s). x_s is guaranteed to be a double root (or higher order root) of $D(x,y)=0$ for $y=y_s$. Such a double root (or higher order root) implies absolute instability if

- y_s has a negative imaginary component and
- Taking the imaginary component of y from y_s to minus infinity, at least a pair of x root of $D(x,y)=0$ splits from the double (or multiple) root x_s to opposite imaginary infinites.

Specifically, the absolutely unstable solutions will have a normalized growth rate of $Im(y_s)$.

We apply numerical methods to check the Briggs-Bers criterion and then analytically find the threshold value of ϵ for the onset of absolute instability. The details of the derivation are given in the Appendix. We summarize the results below.

We confirmed the well-known result that absolute instability always occurs when the beam and circuit modes intersect at a negative group velocity point ($\frac{kL}{\pi}\notin[1,2]$ for the example in Fig. 1) for all nonzero beam current. For interactions with positive group velocity ($kL/\pi\in(1,2)$), the absolute instability is found for both the upper and lower band edges. The normalized threshold current is analytically calculated as

$$\epsilon_{lower}=\left(\frac{u\Big((1-u)^2-2\delta(1-u)\Big)}{2\rho}\right)^2, \tag{4}$$

for the lower band edge and as

$$\begin{aligned}\epsilon_{upper}&=-\frac{27}{256}\kappa^4u^2\rho^2\\ \kappa&=\frac{-8u(\delta+u-1)+2\sqrt{16\delta^2u^2-2\rho^2\Big((u-1)^2+2\delta(u-1)\Big)}}{8u^2+\rho^2},\end{aligned} \tag{5}$$

for the upper band edge.

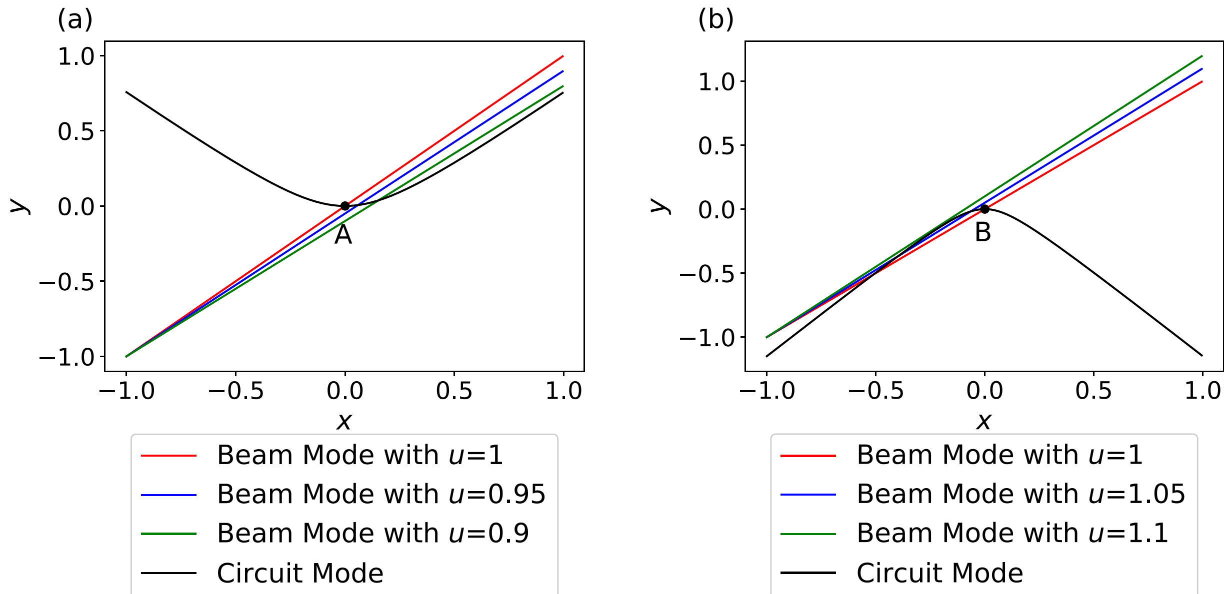

FIG. 2. Normalized dispersion relation for the circuit mode and beam mode at different beam velocities for (a) left, lower band edge, and (b) right, upper band edge.

Figure 3 shows that the threshold ε for the upper band edge is orders of magnitude smaller than it is for the lower band edge, for a similar deviation of the operating point, Q, from points A and B in Fig. 1. This implies that the upper band edge is significantly more prone to absolute instability than the lower band edge. While the dispersion relation (1) for the upper and lower band edges is normalized differently, for the same current, the value of ε typically differs only by a factor of order unity between the lower and upper band edges.

III. TEMPORAL EVOLUTION OF GREEN'S FUNCTION

When the TWT is not subjected to an absolute instability, an initial perturbation may still undergo transient growth (at a fixed position z) before the perturbation is convected away. There will also be transient growth in the perturbations before simple exponential growth when an absolute instability is present. The Green's function, which is the response to an impulse excitation at t = 0 and z = 0, would

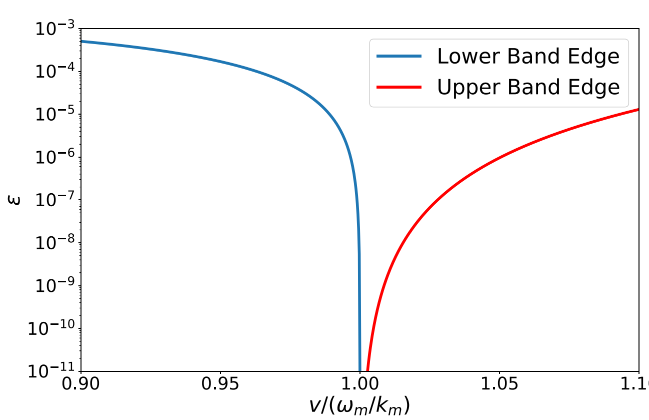

FIG. 3. Threshold values of ϵ for the lower band edge ($v < \omega_m/k_m$) and upper band edge ($v > \omega_m/k_m$). The phase velocity of the circuit mode at either band edge is ω_m/k_m.

show both properties.[9,10] From the dispersion relation, Eq. (1), the Green's function may be constructed[9–11,14]

$$G(z,t) = \int d\omega \int dk \frac{e^{i(\omega t - kz)}}{D(\omega,k)} \propto \int d\omega\, e^{i(\omega t - k(\omega)z)}, \tag{6}$$

where $k(\omega)$ is the solution to $D(\omega, k) = 0$. In the evaluation of the Green's function, Eq. (6), it is found that the dominant term in the exponents of the asymptotic expansion of the last integral of Eq. (6) gives an adequate approximation for the temporal evolution, including exponentiation at the fractional power of t.[11,14,17] Thus, we use the saddle point method and approximate Eq. (6) as

$$G(z,t) \propto e^{i(\omega_s t - k_s z)}, \tag{7}$$

where (ω_s, k_s) is the meaningful root that satisfies

$$D(\omega_s, k_s) = 0,$$
$$\left(\frac{\partial D}{\partial \omega} + \frac{t}{z}\frac{\partial D}{\partial k}\right)\bigg|_{\omega_s,k_s} = 0.$$

We express the magnitude of Eq. (7) in exponential form, $\exp[k_m z f(T)]$, where $T = \frac{\omega_m t}{k_m z}$. Figure 4(a) shows the time dependence of $f(T)$ for the lower band edge when it is stable, marginally stable, and unstable against absolute instability. In all three cases, $f(T) \sim T^{1/3}$, transiently [Fig. 4(a)]. The same is true in the upper band edge, as shown in Fig. 4(b). Note that these explicit calculations of the Green function validated the stability criterion [Eqs. (4) and (5)]. Figures 4(a) and 4(b) show the transition from the transient exponentiation at a fractional power of t (at fixed z) to simple exponential growth when there is an absolute instability in the sense of Briggs and Bers. When the TWT is not subjected to an absolute instability, an initial perturbation may still undergo transient growth (at a fixed position z) before the

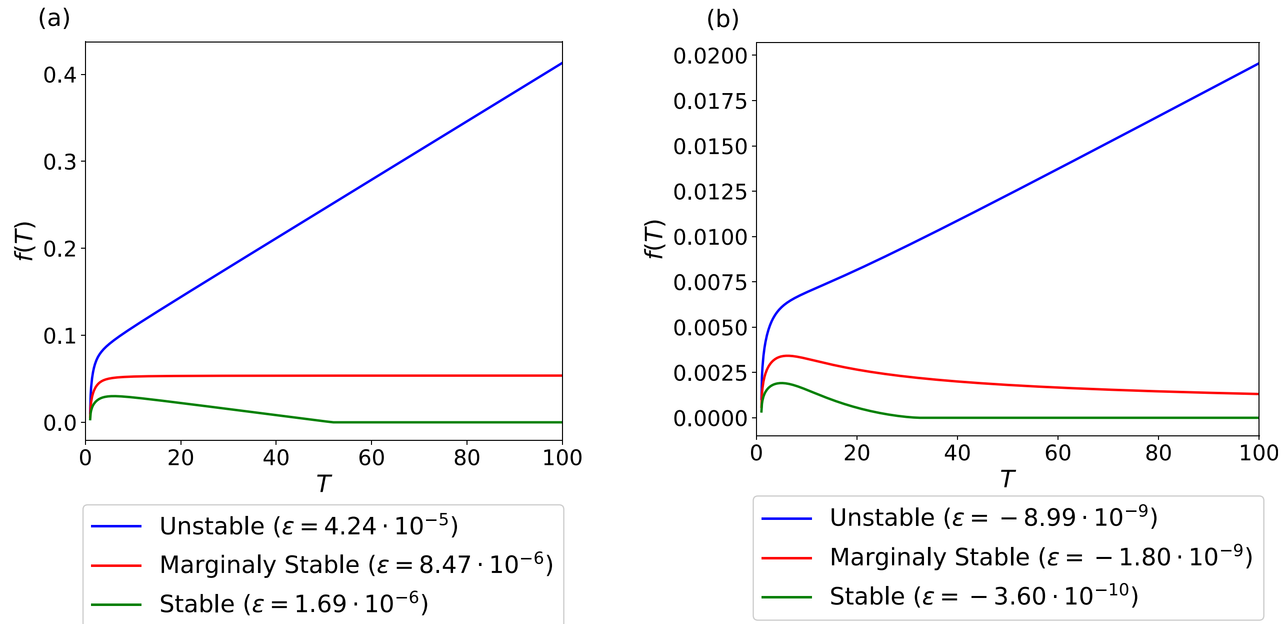

FIG. 4. f(T) for (a) left, the lower band edge, with $v = \frac{0.99\omega_m}{k_m}$, (b) right, upper band edge, with $v = 1.01\omega_m/k_m$. Note that ϵ is roughly $2C^3$, where C is the gain parameter. A comparison of ϵ at marginal stability between the two shows that the upper band edge is more susceptible to absolute instability than the lower band edge.

perturbation is convected away, as also shown in Figs. 4(a) and 4(b).

Alternatively, we can express Eq. (7) in exponential form, $\exp[\omega_m t\, g(Z)]$, where $Z = \frac{k_m z}{\omega_m t}$ and $g(Z)$, as shown in Fig. 5. As seen from Figs. 5(a) and 5(b), $g(Z)$ will have very similar shapes for unstable and stable cases and for both lower and upper band edges. These two figures essentially give the spatial distribution of the logarithm of the Green function as given by Eq. (7), from the beam head ($Z = 1$ or $\frac{z}{t} = \frac{\omega_m}{k_m} \cong v$) to the beam tail ($Z = 0$) in a continuous beam. The key difference between stability and instability lies close to $Z = 0$, i.e., as $t \to \infty$. Thus, the cases where $g(0) > 0$ correspond to absolute instability, whereas the cases where $g(Z) \leq 0$ for some $Z < Z_0$ correspond to the absence of absolute instability. Note that the lower band edge has larger exponents in the Green's function; it is caused by the much higher values of ϵ which are required to excite an absolute instability.

IV. CONCLUDING REMARKS

In this paper, we show that the Green's function, at a fixed position z, exponentiates transiently at a rate proportional to $t^{1/3}$, when the beam mode intersects the circuit

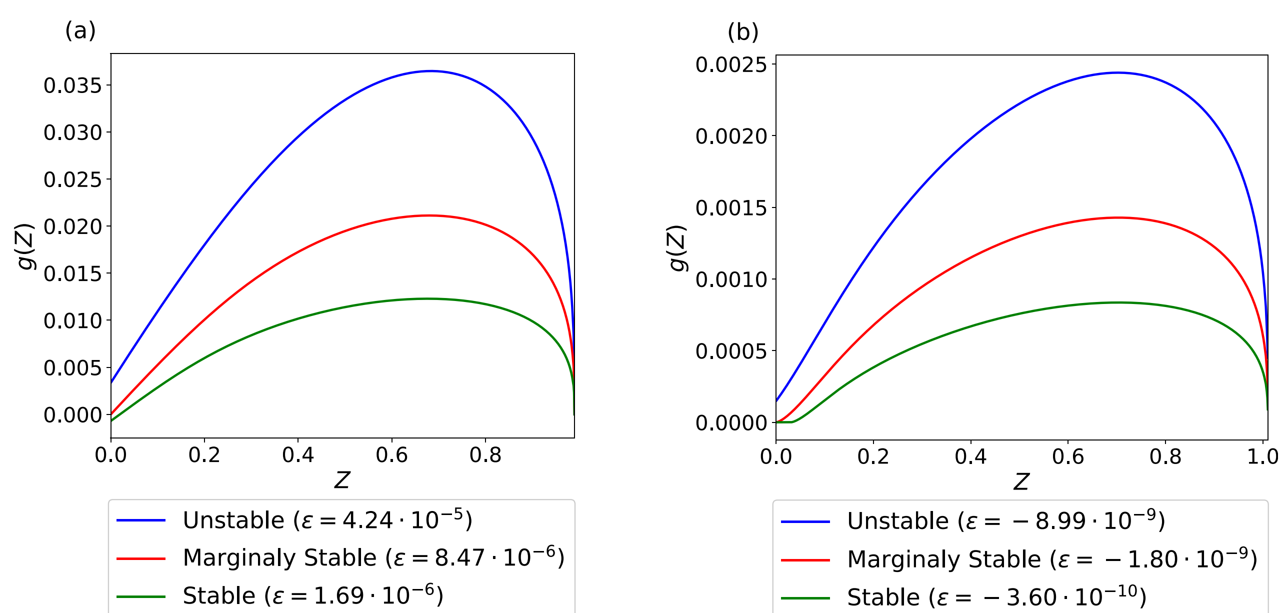

FIG. 5. $g(Z)$ for three cases according to the Briggs-Bers criterion: unstable, marginally stable, and stable. (a) Left, the lower band edge, with $v = 0.99\omega_m/k_m$ and (b) right, the upper band edge, with $v = 1.01\omega_m/k_m$.

mode at a point very close to the band edge, regardless of whether there is an absolute instability or not. This statement applies to both upper and lower band edges, and the transient growth will turn into simple exponential growth if there is an absolute instability but will decay in time if absolute instability is absent. This may be understood as follows: Very close to the band edge, the dispersion relation for the circuit mode behaves like a straight line, $\omega - \omega_m = 0$, i.e., it has zero group velocity, and the beam-circuit interaction may then be described by, when written in the form of Ref. 14,

$$D(\omega, k) \equiv (\omega - kv)^2\left[\omega^2 - \omega_m^2\right] - \omega_m^4\epsilon = 0\,, \qquad (8a)$$

whose Green's function indeed exponentiates as $t^{1/3}$, first discovered by Panofsky and Bander.[11]

If one still assumes operation very close to the band edge, so that the assumption of zero group velocity still holds (i.e., $\omega - \omega_m = 0$) but includes the "space charge effects" in the beam mode, Eq. (8a) is modified to read,[15,16]

$$D(\omega, k) \equiv \left[(\omega - kv)^2 - \omega_q^2\right]\left[\omega^2 - \omega_m^2\right] - \omega_m^4\epsilon = 0. \qquad (8b)$$

In Eq. (8b), ω_q^2 is the square of plasma frequency that includes the plasma reduction factor, and it represents the space charge effect, QC, in Pierce theory of the traveling wave tube.[3,15,16] The Green's function to the dispersion relation of the form [Eq. (8b)] was studied in great detail in Ref. 14. which always shows transient growth at a fractional power of time. This transient growth could be different from $t^{1/3}$, depending on the magnitude of ω_q^2, ϵ, and z.[14] Thus, including the space charge effects of a TWT, very close to the band edge, the Green's function still exponentiates transiently at some fractional power of time, before the asymptotic behavior (at fixed z) predicted by the Briggs-Bers criterion appears.

In summary, when the beam mode intersects with the forward circuit mode of a slow wave structure, an absolute instability exists at both the lower and upper band edges, if the beam current is sufficiently high. The upper band edge is more susceptible to absolute instability than the lower band edge. The threshold condition for the onset of absolute instability at both band edges is given. Close to a band edge, there is always transient growth in the Green's function, at a fractional power of time, whether or not there is an absolute instability.

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APPENDIX: THRESHOLDS OF ABSOLUTE INSTABILITY AT LOWER AND UPPER BAND EDGES

In this Appendix, we outline the derivation of the analytical expressions, Eqs. (4) and (5), the threshold condition for the onset of absolute instability at the lower and upper band edges, respectively.

First, we argue the following points which apply to both the lower and upper band edges. Refer to Eq. (3).

A.1 At marginal stability, $Im(y_s) = 0$.
A.2 There are four pairs of solutions to (x_s, y_s) to (3) which are candidates for absolute instability.
A.3 If a pair of solutions (x_s, y_s) is complex, then $(\overline{x_s}, \overline{y_s})$ is also a pair of solutions, where the bar denotes the complex conjugate.

A.1. This is argued since as mentioned in Sec. II, the normalized growth rate is $Im(y_s)$ and the growth rate at marginal stability is zero.

A.2. To obtain (x_s, y_s), we solve the system of Eq. (3), which becomes

$$(y - ux + 1 - u)^2\left(y^2 + 2\delta y - \rho^2 x^2\right) - \epsilon = 0\,, \qquad (A1)$$

$$2(y - ux + 1 - u)\left(u\left(y^2 + 2\delta y - \rho^2 x^2\right) - \rho^2 x(y - ux + 1 - u)\right) = 0\,. \qquad (A2)$$

This system can then be manipulated into an eighth order polynomial of $x(y)$, eliminating $y(x)$ which has to be equal 0. Therefore, there are eight pairs of solutions (x_s, y_s) to this system. However, there are two branches to the circuit mode hyperbola, $y^2 + 2\delta y - \rho^2 x^2 = 0$. Each branch is responsible for four of the eight solutions. One of the branches is a mathematical artifact of our approximation, and therefore, only four pairs of solutions are candidates for absolute instability.

A.3. Here and in any other case, the statement "(x_s, y_s) is complex" means "x_s has a nonzero imaginary part or y_s has a nonzero imaginary part." The left hand side of (A1) and (A2) can be rewritten as

$$p(x, y) = \sum_i \sum_j a_{i,j} x^i y^j\,, \qquad (A3)$$

with $a_{i,j}$ being the real coefficients. It is easy to show that $p(\bar{x}, \bar{y}) = \overline{p(x, y)}$. Therefore, if (x_s, y_s) is a solution to both (A1) and (A2), then $(\overline{x_s}, \overline{y_s})$ will be a solution also.

We next treat the lower band edge in Sec. 1 and the upper band edge in Sec. 2.

1. Lower band edge

In this section, we will derive the condition for marginal stability for the lower band edge. First, we argue that of the four pairs of solutions that are candidates for absolute instability, two will always be real and two will always be complex. To do so, we use Fig. 6. The upper branch of the hyperbola is the branch that yields solutions that are candidates for absolute instability. The dashed lines represent the solution to $D(x, y) = 0$, and the crosses are the solutions to $\partial D/\partial x = 0$ or $dy/dx = 0$. As ϵ increases, the dashed lines will move away from both the circuit mode and the beam mode. The slope on each dashed line will always be 0 at exactly one point regardless of ϵ. Therefore, there are exactly two real pairs of solutions which, being real, cannot be candidates for absolute instability, and the two remaining pairs of solutions have to be complex, and they are candidates for absolute instability when ϵ is sufficiently large. We believe

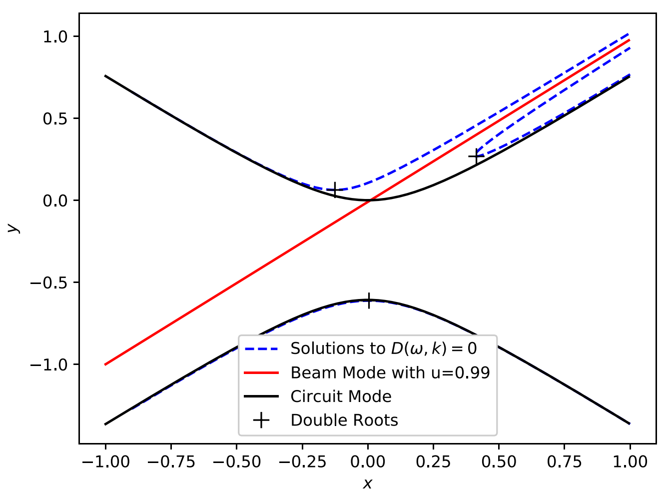

FIG. 6. Graphical representation of solutions to (2) for the lower band edge for $\epsilon = 10^{-3}$.

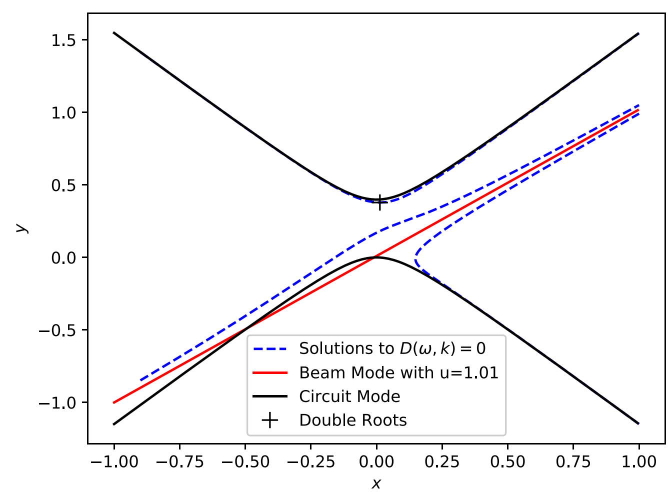

FIG. 7. A graphical representation of solutions to (2) for the upper band edge for $\epsilon = 10^{-3}$ where no real solutions exist around the intersection point.

that Ref. 6 did not check the complex roots for absolute instability for the lower band edge because the existence of real roots is a sufficient condition for stability at the upper band edge (see Sec. 2 of the Appendix). This results in the erroneous claim in Ref. 6 that the lower band edge does not suffer from absolute instability when the beam mode intersects the circuit mode at the forward wave side.

Numerically evaluating the Briggs-Bers criterion for the complex roots does yield absolute instability for sufficiently high values of ϵ. At marginal stability, (x_s, y_s) must be complex and y_s must be real as argued in A.1. Therefore, x_s must have an imaginary part. Furthermore, $(\overline{x_s}, y_s)$ must also be a solution to (3) as argued in A.3. Since (3) guarantees that x_s and $\overline{x_s}$ are double roots to $D(x,y)|_{y_s} = 0$, then

$$D(x,y)|_{y_s} = a(x - x_s)^2(x - \overline{x_s})^2 = a\left(x - 2Re(x_s)x + |x_s|^2\right)^2, \quad \text{(A4)}$$

for some real a. Using (2) to evaluate $D(x,y)|_{y_s}$ and setting each coefficient of the fourth order polynomials of x in Eq. (A4) equal to each other, we obtain a system of five equations with five unknowns (ϵ, $Re(x_s)$, $|x_s|^2$, y_s, and a). Solving this system yields Eq. (4) of the main text.

2. Upper band edge

In this section, we will derive the condition for marginal stability for the upper band edge. First, we argue that out of the four pairs of solutions that are candidates for absolute instability, two will always be complex and two will transition from real to complex as ϵ increases. Figure 7 shows the dispersion relation (2) for a high value of ϵ at which absolute instability exists. Figure 8 shows the dispersion relation (2) for a low value of ϵ at which absolute instability does not exist. In Fig. 7, the bottom branch of the hyperbola is the one that yields solutions that are candidates for absolute instability at the upper band edge. Figure 8 is zoomed in around the intersection point of the beam mode and the circuit mode for a very low value of ϵ, at which there is no absolute instability. Note that this intersection point is to the left of the peak of the circuit mode. For sufficiently high values of ϵ, there are no real solutions to (3) as seen in Fig. 7. As ϵ decreases, both dashed lines in Fig. 7 will move closer to the beam and circuit modes. For sufficiently small values of ϵ, the blue dashed line will fold over the peak of the circuit mode and have a zero derivative ($dy/dx = 0$) at two points as seen in Fig. 8. Therefore, there are two solutions that are always complex with nonzero imaginary parts and two solutions that transition from real to imaginary as ϵ increases.

Numerically evaluating the Briggs-Bers criterion for the two roots that are always complex, in both Figs. 7 and 8, never yields absolute instability. Numerically evaluating the Briggs-Bers criterion for the two roots that transition from real to complex as ϵ increases always yields absolute instability when they are complex. Marginal stability is therefore the point when these two pairs of solutions transition from real to complex. When these solutions are complex, they must have the same real parts as argued in A.3. Therefore, to become real, these two pairs of solutions must coincide on the real axis. For each pair of solutions $D(x,y)|_{y_s}$ must have

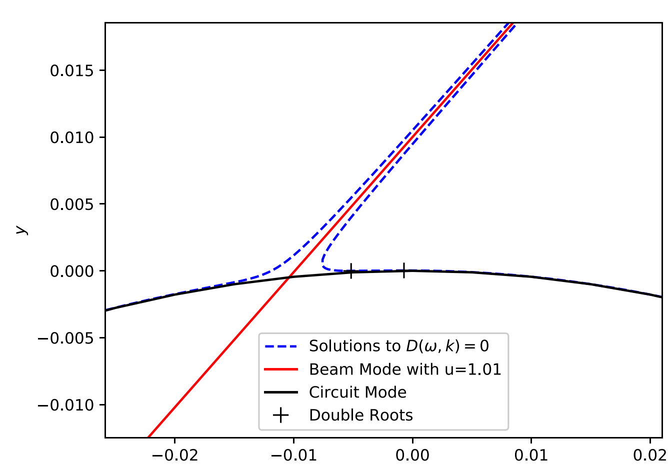

FIG. 8. A graphical representation of solution to (2) for the upper band edge for $\epsilon = 10^{-9}$ where two real solutions exist around the intersection point.

a double x root. However, since these two pairs are the same, $D(x,y)|_{y_s}$ must have two double x roots. This means that $D(x,y)|_{y_s}$ has a triple root in x and therefore can be written as

$$D(x,y)|_{y_s} = a(x-x_s)^3(x-b)\,, \tag{A5}$$

for some real a and b.

This triple root in x at transition to absolute instability may also be seen in Fig. 8, where the two double roots on the blue curve must merge to become a triple root before they disappear as ϵ increases. This appearance of a triple root in x (or in k) is also the threshold condition for the onset of the absolute instability in a gyrotron traveling wave amplifier,[18] because the latter's dispersion relation is very similar to that of TWT near the upper band edge, as noted in Ref. 6.

Using (2) to evaluate $D(x,y)|_{y_s}$ and setting each coefficient of the fourth order polynomials of x in Eq. (A5) equal to each other, we obtain a system of five equations with five unknowns (ϵ, x_s, y_s, a, and b). Solving this system yields Eq. (5) of the main text.

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